



Stability Analysis and Numerical Simulation of 1-D and 2-D Radial Flow towards an Oil Well

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(Received and accepted July 1995)

Abstract—The radial flow of oil towards a well in one and two dimensions is modeled by a family of finite difference schemes. This family depends on one parameter θ , $0 \leq \theta \leq 1$. The stability of the proposed schemes is analyzed applying the matrix method, which takes into account boundary conditions. Particularly, in the 2-D case, an “almost pentadiagonal” matrix is obtained choosing an appropriate order of equations and unknowns. We prove that this matrix may be symmetrized by a similarity transformation. Therefore, studying bounds for the corresponding eigenvalues, unconditional stability is found for $\theta \geq 1/2$ and stability restrictions are established for $\theta < 1/2$. Numerical simulations are presented using the BSOR (Block Successive Over Relaxation) method to solve the resulting system of linear equations. The finite difference solution has perfectly reproduced the analytical solution of a simplified 1-D model.

Keywords—Stability analysis, Finite differences, Oil flow, Simulation.

1. INTRODUCTION

The general equations that govern fluid flow through porous media are obtained by combining Darcy's Law and the equation of conservation of mass [1,2]. Although all real oil reservoirs are three-dimensional, in many practical situations we may assume that flow in some coordinate is negligible. In this paper, single-phase flow of oil towards a well is analyzed in one and two dimensions. As we consider only single-phase flow, a scalar equation is used. This equation has practical interest in the field of reservoir engineering, because it is applied in well test analysis [3]. During the well test, the pressure response to changing production or injection conditions is measured. The reservoir properties characterize that response; therefore, the aim of well testing is to infer those properties analyzing pressure behavior. Traditionally, 1-D radial models with constant

We are indebted to the University of Buenos Aires, the “Consejo Nacional de Investigaciones Científicas y Técnicas de Argentina”, and the oil companies Pluspetrol and Amoco Argentina for partially supporting this work.

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properties have been used in well test analysis [3]. Nevertheless, with a 2-D model (in radial and vertical coordinates) the effects of gravity, vertical permeability and rock heterogeneities can be assessed.

The purpose of this paper is to analyze the stability conditions of a family of finite difference schemes applied to solve the 1-D and 2-D models and to present some numerical simulations. Stability analysis of finite difference schemes for parabolic type equations were described in previous papers [4–6], involving matrices with constant [5,6] or variable coefficients [4]. Nevertheless, only Dirichlet boundary conditions are considered [4,5], or other conditions are included but in a linear case with constant coefficients [6]. Besides, the resulting matrices of the linear systems were symmetric [5] or the eigenvalues may be calculated explicitly [6].

The 2-D equation with mixed boundary conditions presented herewith provides a nonsymmetric, almost pentadiagonal matrix with variable coefficients, and as far as we know, it has not been studied before. We obtain a similarity transformation that symmetrizes it, and then the stability is proved analyzing bounds for the corresponding real eigenvalues.

In the first part of the paper (Sections 2 and 3), the differential equations of the model and their discretization are presented. The crux of this matter is Section 4, where the stability conditions of the finite difference schemes are analyzed. Besides, the BSOR implementation for this particular case is described in Section 5. Finally, numerical simulations are analyzed in Section 6.

2. THE 1-D AND 2-D MODELS

We assume a multilayer, cylindrical reservoir that is enclosed at the top, bottom and outer radius by an impermeable boundary. It has a well located at its central axis that penetrates the formation completely. This geometric model is shown in Figure 1. The well and outer radius are r_w and r_e , respectively, and H is the reservoir thickness. Obviously, cylindrical coordinates (r, θ, z) are used. The reservoir contains only oil, which is a slightly compressible fluid of constant compressibility and constant viscosity.

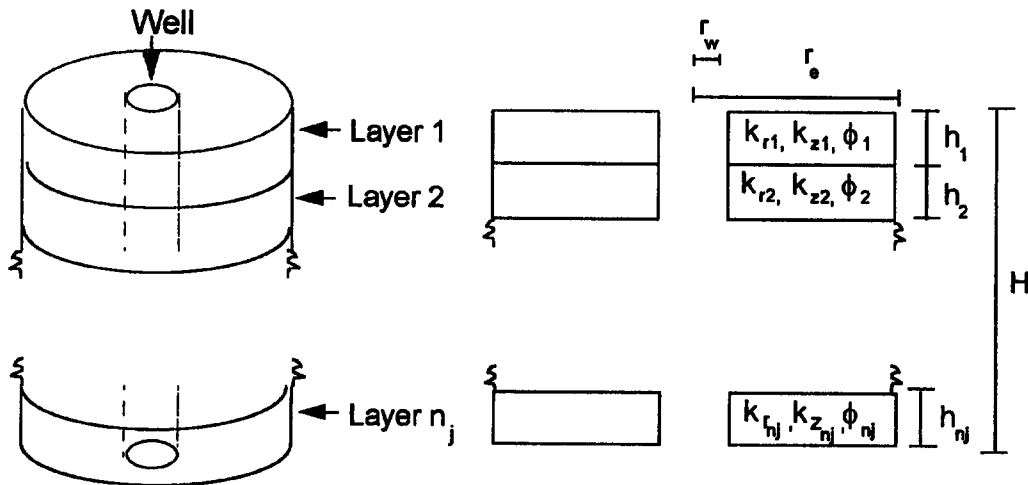


Figure 1. Reservoir model.

1-D Model

If the following assumptions are also stated:

- all layers have the same properties,
- there is no vertical flow,
- the solution domain is axisymmetric and all rock properties are functions of (r) and boundary conditions are functions of (r, t) ,

the only coordinate relevant to flow is the radius r and, therefore, a 1-D model is obtained,

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \rho(p) k(r) \frac{\partial p}{\partial r}(r, t) \right) + q(r, t) = \phi(r) \mu c \rho(p) \frac{\partial p}{\partial t}(r, t), \quad (1)$$

where radial distance, r , and time, t , are the independent variables and pressure, $p(r, t)$, is the unknown. The parameters are viscosity, μ , compressibility, c , absolute permeability, $k(r)$, and porosity, $\phi(r)$. The oil density is $\rho(p)$. Neglecting source or sink term, $q(r, t)$, and also gradient squared terms, equation (1) becomes

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r k(r) \frac{\partial p}{\partial r}(r, t) \right) = \phi(r) \mu c \frac{\partial p}{\partial t}(r, t). \quad (2)$$

Equation (2) is a parabolic partial differential equation. The initial condition is the known reservoir pressure p_{IN} ,

$$p(t = 0, r) = p_{IN}. \quad (3)$$

The boundary condition at the well is

$$k(r) r \frac{\partial p}{\partial r}(r, t) \Big|_{r=r_w} = \frac{q(t) \mu}{2\pi H}, \quad t > 0, \quad (4)$$

where the flow rate q may vary with time.

The outer boundary condition assumes absence of flow in the outer radius r_e ,

$$r \frac{\partial p}{\partial r}(r, t) \Big|_{r=r_e} = 0, \quad t > 0. \quad (5)$$

If permeability and porosity are constant, there is an analytical solution [7] for equation (2) with initial and boundary conditions (3), (4) and (5). This solution is given by a series expansion whose terms include Bessel functions. If permeability and porosity are not constant, an analytical solution exists only for special cases; for instance, when permeability is an arbitrary function of position with small variations from a mean value and porosity is constant [8]. However, in many cases the problem cannot be analytically solved and a numerical solution is required.

2-D Model

Maintaining only the third assumption of the 1-D model, i.e.,

- the solution domain is axisymmetric and all rock properties are functions of (r, Z) and boundary conditions are functions of (r, Z, t) ,

a model in two dimensions, radial and vertical (r, Z) , is obtained,

$$\begin{aligned} \frac{1}{r} \frac{\partial}{\partial r} \left(r \rho(p) \frac{k_r(r, Z)}{\mu} \frac{\partial p}{\partial r}(r, Z, t) \right) + \frac{\partial}{\partial Z} \left(\rho(p) \frac{k_z(r, Z)}{\mu} \left(\frac{\partial p}{\partial Z}(r, Z, t) - g \rho(p) \right) \right) \\ + \hat{q}(r, Z, t) = \rho(p) \phi(r, Z) c \frac{\partial p}{\partial t}(r, Z, t). \end{aligned} \quad (6)$$

The pressure-volume-temperature (PVT) behavior of the fluid may be expressed in terms of the formation volume factor, B_0 , defined as the volume of oil at reservoir pressure divided by the volume of oil at standard condition (SC),

$$B_0(p) = \frac{V_0(p)}{[V_0]_{SC}}, \quad \text{that is} \quad \rho(p) = \frac{\rho_{SC}}{B_0(p)}.$$

Replacing $\rho(p)$ in equation (6) and dividing by ρ_{SC} , it becomes [1],

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r T_r(r, \mathcal{Z}, p) \frac{\partial p}{\partial r}(r, \mathcal{Z}, t) \right) + \frac{\partial}{\partial \mathcal{Z}} \left(T_z(r, \mathcal{Z}, p) \left(\frac{\partial p}{\partial \mathcal{Z}}(r, \mathcal{Z}, t) - g \frac{\rho_{SC}}{B_0(p)} \right) \right) + q(r, \mathcal{Z}, t) = \beta(r, \mathcal{Z}, p) \frac{\partial p}{\partial t}(r, \mathcal{Z}, t), \quad (7)$$

where

$$T_r(r, \mathcal{Z}, p) = \frac{k_r(r, \mathcal{Z})}{\mu B_0(p)}, \quad T_z(r, \mathcal{Z}, p) = \frac{k_z(r, \mathcal{Z})}{\mu B_0(p)}, \quad \beta(r, \mathcal{Z}, p) = \frac{\phi(r, \mathcal{Z})c}{B_0(p)}, \quad q(r, \mathcal{Z}, t) = \frac{\hat{q}(r, \mathcal{Z}, t)}{\rho_{SC}}.$$

In a first approach, we consider $B_0(p)$ approximately constant to obtain a linear partial differential equation. Moreover, the flow rate is considered in the boundary conditions. The boundary conditions are generalizations of those stated in the 1-D model while the initial condition is the same. Therefore, the problem to be solved is

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r T_r(r, \mathcal{Z}) \frac{\partial p}{\partial r}(r, \mathcal{Z}, t) \right) + \frac{\partial}{\partial \mathcal{Z}} \left(T_z(r, \mathcal{Z}) \left(\frac{\partial p}{\partial \mathcal{Z}}(r, \mathcal{Z}, t) - g \frac{\rho_{SC}}{B_0} \right) \right) = \beta(r, \mathcal{Z}) \frac{\partial p}{\partial t}(r, \mathcal{Z}, t), \quad (8)$$

with initial condition

$$p(r, \mathcal{Z}, t = 0) = p_{IN}. \quad (9)$$

The boundary conditions at the well ($r = r_w$) are

- known total flow rate,

$$q_T(t) = \frac{2\pi}{\mu B_0} \int_0^H k_r(r_w, \mathcal{Z}) \left(r \frac{\partial p}{\partial r} \right)_{r=r_w} d\mathcal{Z}, \quad (10)$$

- pressures related by gravity forces,

$$p(r_w, \mathcal{Z}, t) = p_{wf} + \rho g \mathcal{Z}, \quad (11)$$

where $p_{wf} = p(r_w, \mathcal{Z} = 0, t)$.

At reservoir boundaries (outer, top and bottom), the nonflow condition is imposed. By Darcy's Law,

$$\frac{k_r(r, \mathcal{Z})}{\mu} \left(\frac{\partial p}{\partial r}(r, \mathcal{Z}, t) \right) \Big|_{r=r_e} = 0, \quad (12)$$

$$\frac{k_z(r, \mathcal{Z})}{\mu} \left(\frac{\partial p}{\partial \mathcal{Z}}(r, \mathcal{Z}, t) - g \frac{\rho_{SC}}{B_0} \right) \Big|_{\mathcal{Z}=0, H} = 0. \quad (13)$$

Adding different assumptions, approximate analytical solutions may be obtained [9,10]. Nevertheless, the general case requires a numerical solution.

3. NUMERICAL APPROACH

We describe now the numerical approximation of the 2-D model, equations (8)–(13). In a similar way the 1-D discretization is deduced [11]. In order to work with dimensionless variables, we define

$$x = \ln \left(\frac{r}{r_w} \right), \quad z = \frac{\mathcal{Z}}{H}, \quad t = \frac{t}{\mu c}. \quad (14)$$

The aim of introducing the variable x is to obtain a grid with more points near the wellbore using a constant increment Δx . Replacing in equation (8), we get

$$\frac{1}{r_w^2} e^{-2x} \frac{\partial}{\partial x} \left(T_x(x, z) \frac{\partial p}{\partial x}(x, z, t) \right) + \frac{1}{H} \frac{\partial}{\partial z} \left(T_z(x, z) \left(\frac{1}{H} \frac{\partial p}{\partial z}(x, z, t) - g \frac{\rho_{SC}}{B_0} \right) \right) = \frac{\beta(x, z)}{\mu c} \frac{\partial p}{\partial t}(x, z, t). \quad (15)$$

We consider a rectangular domain in the x, z plane, $\Omega = \{(x, z) / 0 \leq x \leq \ln(r_e/r_w), 0 \leq z \leq 1\}$. It is divided into a uniform grid with grid points $(x_i, z_j) = (i\Delta x, (j + (1/2))\Delta z)$, $0 \leq i \leq n_i$, $0 \leq j \leq n_j$, where Δx and Δz are the spatial increments in the x and z direction, respectively. We use discretized times $t^n = n\Delta t$, Δt being the time discretization step. Pressure at the representative mesh point (x_i, z_j, t^n) is denoted by $P_{i,j}^n$. Then, equation (15) is discretized using a family of finite difference schemes, depending upon a parameter θ , $0 \leq \theta \leq 1$,

$$\frac{\beta_{i,j}^{n+1}}{\mu c} \left(\frac{P_{i,j}^{n+1} - P_{i,j}^n}{\Delta t} \right) = \theta \Phi_{i,j}^{n+1} + (1 - \theta) \Phi_{i,j}^n, \quad (16)$$

where

$$\Phi_{i,j}^n = \frac{1}{r_w^2} e^{-2x_i} \left(\frac{T_{x_{i+(1/2),j}} (P_{i+1,j}^n - P_{i,j}^n) / \Delta x - T_{x_{i-(1/2),j}} (P_{i,j}^n - P_{i-1,j}^n) / \Delta x}{\Delta x} \right) + \frac{1}{H} \left(\frac{T_{z_{i,j+(1/2)}} ((1/H) (P_{i,j+1}^n - P_{i,j}^n) / \Delta z - g(\rho_{SC}/B_0)) - T_{z_{i,j-(1/2)}} ((1/H) (P_{i,j}^n - P_{i,j-1}^n) / \Delta z - g(\rho_{SC}/B_0))}{\Delta z} \right). \quad (17)$$

Equation (16) may be written as

$$\begin{aligned} \theta c_{i,j} P_{i-1,j}^{n+1} + \theta g_{i,j} P_{i,j-1}^{n+1} + (1 + \theta a_{i,j}) P_{i,j}^{n+1} + \theta f_{i,j} P_{i,j+1}^{n+1} + \theta b_{i,j} P_{i+1,j}^{n+1} \\ = -(1 - \theta) c_{i,j} P_{i-1,j}^n - (1 - \theta) g_{i,j} P_{i,j-1}^n + (1 - (1 - \theta) a_{i,j}) P_{i,j}^n \\ - (1 - \theta) f_{i,j} P_{i,j+1}^n - (1 - \theta) b_{i,j} P_{i+1,j}^n + e_{i,j}, \end{aligned} \quad (18)$$

defining

$$\begin{aligned} c_{i,j} &= -\frac{\mu c \Delta t}{\beta_{i,j}} \frac{e^{-2x_i}}{r_w^2 \Delta x^2} T_{x_{i-1/2,j}}, & b_{i,j} &= -\frac{\mu c \Delta t}{\beta_{i,j}} \frac{e^{-2x_i}}{r_w^2 \Delta x^2} T_{x_{i+1/2,j}}, \\ g_{i,j} &= -\frac{\mu c \Delta t}{\beta_{i,j}} \frac{1}{H^2 \Delta z^2} T_{z_{i,j-1/2}}, & f_{i,j} &= -\frac{\mu c \Delta t}{\beta_{i,j}} \frac{1}{H^2 \Delta z^2} T_{z_{i,j+1/2}}, \\ a_{i,j} &= -[c_{i,j} + b_{i,j} + g_{i,j} + f_{i,j}], & e_{i,j} &= \frac{\mu c \Delta t}{\beta_{i,j}} \frac{g \rho_{SC}}{H \Delta z B_0} (T_{z_{i,j-1/2}} - T_{z_{i,j+1/2}}), \end{aligned}$$

$1 \leq i \leq n_i - 1, 1 \leq j \leq n_j - 1$.

Boundary conditions at the well ($i = 0$) are given by equations (10) and (11).

- Equation (10) is discretized as

$$\begin{aligned} q_T &= \frac{2\pi}{\mu B_0} \sum_{j=0}^{n_j} h_j k_{r0,j} \left[\theta \left(\frac{P_{0,j}^{n+1} - P_{-1,j}^{n+1}}{\Delta x} \right) + (1 - \theta) \left(\frac{P_{0,j}^n - P_{-1,j}^n}{\Delta x} \right) \right] \\ &= \frac{2\pi H \Delta z}{\mu B_0 \Delta x} \sum_{j=0}^{n_j} k_{r0,j} [\theta (P_{0,j}^{n+1} - P_{-1,j}^{n+1}) + (1 - \theta) (P_{0,j}^n - P_{-1,j}^n)], \end{aligned} \quad (19)$$

where $P_{-1,j}$ are auxiliary points outside the boundary.

- Equation (11) is discretized as

$$P_{0,j}^n = P_{0,0}^n + \rho g j H \Delta z, \quad \forall j = 1, n_j. \quad (20)$$

At the outer boundary ($i = n_i, j = 0, j = n_j$), equations (12) and (13) are satisfied taking

$$T_{x_{n_i+1/2,j}} = 0, \quad T_{z_{i,-1/2}} = 0, \quad T_{z_{i,n_j+1/2}} = 0, \quad j = 0, \dots, n_j, \quad i = 0, \dots, n_i. \quad (21)$$

4. STABILITY ANALYSIS

In order to obtain a convenient structure of the resulting linear system, the following order of unknowns and equations is chosen:

- unknowns $((n_j + 1) \cdot (n_i + 1) + 1)$:

$$P_{-1,0}^{n+1}, P_{-1,1}^{n+1}, \dots, P_{-1,n_j}^{n+1}; P_{0,0}^{n+1}; P_{1,0}^{n+1}, \dots, P_{1,n_j}^{n+1}; P_{2,0}^{n+1}, \dots, P_{2,n_j}^{n+1}; \dots; P_{n_i,0}^{n+1}, \dots, P_{n_i,n_j}^{n+1}.$$

- equations $((n_j + 1) \cdot (n_i + 1) + 1)$:
 1. equation (18) corresponding to $i = 0, j = 0, \dots, n_j$; $((n_j + 1)$ equations),
 2. equation (19); (1 equation),
 3. equation (18) corresponding to $((j = 0, \dots, n_j); i = 1, \dots, n_i)$; $((n_j + 1) \cdot n_i$ equations).

Therefore, the linear system in matrix form is

$$M_1 \begin{pmatrix} P_{-1,0} \\ P_{-1,n_j} \\ P_{0,0} \\ P_{1,0} \\ P_{1,j} \\ P_{1,n_j} \\ P_{n_i,0} \\ P_{n_i,n_j} \end{pmatrix}^{n+1} = \begin{pmatrix} d_0 \\ \vdots \\ d_{n_j} \\ \frac{q_T B_0 \mu \Delta x}{2\pi H \Delta z} - \rho g H \Delta z \sum_{j=0}^{n_j} j k_{r0,j} \\ 0 \\ -c_{1,j} g \rho j H \Delta z \\ -c_{1,n_j} g \rho H \\ 0 \\ 0 \\ 0 \end{pmatrix} + M_2 \begin{pmatrix} P_{-1,0} \\ P_{-1,n_j} \\ P_{0,0} \\ P_{1,0} \\ P_{1,j} \\ P_{1,n_j} \\ P_{n_i,0} \\ P_{n_i,n_j} \end{pmatrix}^n + \begin{pmatrix} e_{0,0} \\ e_{0,n_j} \\ 0 \\ e_{1,0} \\ e_{1,j} \\ e_{1,n_j} \\ e_{n_i,0} \\ e_{n_i,n_j} \end{pmatrix}, \quad (22)$$

where $d_j = \rho g H \Delta z ((g_{0,j} - f_{0,j}) + j(c_{0,j} + b_{0,j}))$, $j = 0, n_j$, and $M_1, M_2 \in R^{N \times N}$ with $N = (n_j + 1) \cdot (n_i + 1) + 1$. It is even possible to improve matrix M_1 structure and to reduce the number of linear equations to be solved. To do this, let us replace the $(n_j + 2)$ row of M_1 (corresponding to equation (19)) by $(\text{row})_{n_j+2} \leftarrow (\text{row})_{n_j+2} + \sum_{j=0}^{n_j} (k_{r0,j}/c_{0,j}) (\text{row})_j$. Then, equation (22) becomes

$$\bar{M}_1 \begin{pmatrix} P_{-1,0} \\ P_{-1,n_j} \\ P_{0,0} \\ P_{1,0} \\ P_{1,n_j} \\ P_{n_i,0} \\ P_{n_i,n_j} \end{pmatrix}^{n+1} = \begin{pmatrix} d_0 \\ \vdots \\ d_{n_j} \\ \frac{q_T B_0 \mu \Delta x}{2\pi H \Delta z} - \rho g H \Delta z \sum_{j=0}^{n_j} j k_{r0,j} + \sum_{j=0}^{n_j} \frac{k_{r0,j}}{c_{0,j}} d_j \\ 0 \\ \vdots \\ -c_{1,n_j} g \rho H \\ 0 \\ 0 \\ 0 \end{pmatrix} + \bar{M}_2 \begin{pmatrix} P_{-1,0} \\ P_{-1,n_j} \\ P_{0,0} \\ P_{1,0} \\ P_{1,n_j} \\ P_{n_i,0} \\ P_{n_i,n_j} \end{pmatrix}^n + \begin{pmatrix} e_{0,0} \\ e_{0,n_j} \\ \sum_{j=0}^{n_j} \frac{k_{r0,j}}{c_{0,j}} e_{0,j} \\ e_{1,0} \\ e_{1,n_j} \\ e_{n_i,0} \\ e_{n_i,n_j} \end{pmatrix}, \quad (23)$$

where \overline{M}_1 is

$$\begin{pmatrix} C_0 & \varphi & B_0 & 0 & 0 & 0 & 0 \\ \mathbf{0}^t & \eta & \zeta^t & \mathbf{0}^t & \mathbf{0}^t & \mathbf{0}^t & \mathbf{0}^t \\ 0 & \mathbf{c} & A_1 & B_1 & 0 & 0 & 0 \\ 0 & \mathbf{0} & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & \mathbf{0} & 0 & C_1 & A_i & B_i & 0 \\ 0 & \mathbf{0} & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & \mathbf{0} & 0 & 0 & 0 & C_{n_i} & A_{n_i} \end{pmatrix}$$

with

$$C_i = \begin{pmatrix} \theta c_{i,0} & 0 & 0 & 0 & 0 \\ 0 & \theta c_{i,1} & 0 & 0 & 0 \\ 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & 0 & \theta c_{i,n_j} \end{pmatrix}, \quad A_i = \begin{pmatrix} 1 + \theta a_{i,0} & \theta f_{i,0} & 0 & 0 & 0 \\ \theta g_{i,1} & 1 + \theta a_{i,1} & \theta f_{i,1} & 0 & 0 \\ 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \theta g_{i,n_j} & 1 + \theta a_{i,n_j} \end{pmatrix},$$

$$B_i = \begin{pmatrix} \theta b_{i,0} & 0 & 0 & 0 & 0 \\ 0 & \theta b_{i,1} & 0 & 0 & 0 \\ 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & 0 & \theta b_{i,n_j} \end{pmatrix}, \quad \varphi = \begin{pmatrix} 1 + \theta \overline{a_{0,0}} \\ 1 + \theta \overline{a_{0,1}} \\ \cdot \\ \cdot \\ 1 + \theta \overline{a_{0,n_j}} \end{pmatrix}, \quad \eta = \sum_{j=0}^{n_j} \frac{k_{r_{0,j}}}{c_{0,j}} + \theta \sum_{j=0}^{n_j} k_{r_{0,j}} \left(1 + \frac{\overline{a_{0,j}}}{c_{0,j}} \right),$$

$$\zeta^t = \left(\theta \frac{k_{r_{0,0}}}{c_{0,0}} b_{0,0}, \theta \frac{k_{r_{0,1}}}{c_{0,1}} b_{0,1}, \dots, \theta \frac{k_{r_{0,n_j}}}{c_{0,n_j}} b_{0,n_j} \right), \quad \mathbf{c} = \begin{pmatrix} \theta c_{1,0} \\ \theta c_{1,1} \\ \cdot \\ \cdot \\ \theta c_{1,n_j} \end{pmatrix};$$

and \overline{M}_2 is

$$\begin{pmatrix} \tilde{C}_0 & \tilde{\varphi} & \tilde{B}_0 & 0 & 0 & 0 & 0 \\ \mathbf{0}^t & \tilde{\eta} & \tilde{\zeta}^t & \mathbf{0}^t & \mathbf{0}^t & \mathbf{0}^t & \mathbf{0}^t \\ 0 & \tilde{\mathbf{c}} & \tilde{A}_1 & \tilde{B}_1 & 0 & 0 & 0 \\ 0 & \mathbf{0} & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & \mathbf{0} & 0 & \tilde{C}_1 & \tilde{A}_i & \tilde{B}_i & 0 \\ 0 & \mathbf{0} & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & \mathbf{0} & 0 & 0 & 0 & \tilde{C}_{n_i} & \tilde{A}_{n_i} \end{pmatrix}$$

with

$$\tilde{C}_i = \begin{pmatrix} -(1-\theta)c_{i,0} & 0 & 0 & 0 & 0 \\ 0 & -1(1-\theta)c_{i,1} & 0 & 0 & 0 \\ 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & 0 & -(1-\theta)c_{i,n_j} \end{pmatrix},$$

$$\tilde{A}_i = \begin{pmatrix} 1 - (1-\theta)a_{i,0} & -(1-\theta)f_{i,0} & 0 & 0 & 0 \\ -(1-\theta)g_{i,1} & 1 - (1-\theta)a_{i,1} & -(1-\theta)f_{i,1} & 0 & 0 \\ 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & -(1-\theta)g_{i,n_j} & 1 - (1-\theta)a_{i,n_j} \end{pmatrix},$$

$$\tilde{B}_i = \begin{pmatrix} -(1-\theta)b_{i,0} & 0 & 0 & 0 & 0 \\ 0 & -(1-\theta)b_{i,1} & 0 & 0 & 0 \\ 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & 0 & -(1-\theta)b_{i,n_j} \end{pmatrix}, \quad \tilde{\varphi} = \begin{pmatrix} 1 - (1-\theta)\overline{a_{0,0}} \\ 1 - (1-\theta)\overline{a_{0,1}} \\ \cdot \\ \cdot \\ 1 - (1-\theta)\overline{a_{0,n_j}} \end{pmatrix},$$

$$\tilde{\eta} = \sum_{j=0}^{n_j} \frac{k_{r0,j}}{c_{0,j}} - (1-\theta) \sum_{j=0}^{n_j} k_{r0,j} \left(1 + \frac{\overline{a_{0,j}}}{c_{0,j}}\right),$$

$$\tilde{\zeta}^t = \left(-(1-\theta) \frac{k_{r0,0}}{c_{0,0}} b_{0,0}, -(1-\theta) \frac{k_{r0,1}}{c_{0,1}} b_{0,1}, \dots, -(1-\theta) \frac{k_{r0,n_j}}{c_{0,n_j}} b_{0,n_j} \right), \quad \tilde{c} = \begin{pmatrix} -(1-\theta)c_{1,0} \\ -(1-\theta)c_{1,1} \\ \vdots \\ -(1-\theta)c_{1,n_j} \end{pmatrix}$$

Consequently, system (23) may be split into two systems of equations, i.e.,

$$M_{a1} \begin{pmatrix} P_{0,0} \\ P_{1,0} \\ P_{1,n_j} \\ P_{n_i,0} \\ P_{n_i,n_j} \end{pmatrix}^{n+1} = \begin{pmatrix} \frac{qT B_0 \mu \Delta x}{2\pi H \Delta z} - \rho g H \Delta z \sum_{j=0}^{n_j} j k_{r0,j} + \sum_{j=0}^{n_j} \frac{k_{r0,j}}{c_{0,j}} d_j \\ 0 \\ -c_{1,n_j} g \rho H \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (24)$$

$$+ M_{a2} \begin{pmatrix} P_{0,0} \\ P_{1,0} \\ P_{1,n_j} \\ P_{n_i,0} \\ P_{n_i,n_j} \end{pmatrix}^n + \begin{pmatrix} \sum_{j=0}^{n_j} \frac{k_{r0,j}}{c_{0,j}} e_{0,j} \\ e_{1,0} \\ e_{1,n_j} \\ e_{n_i,0} \\ e_{n_i,n_j} \end{pmatrix},$$

where M_{a1} and M_{a2} are obtained from $\overline{M_1}$ and $\overline{M_2}$ eliminating the first $(n_j + 1)$ rows and the first $(n_j + 1)$ columns.

Once $(P_{0,0}, P_{1,0}, \dots, P_{1,n_j}, \dots, P_{n_i,0}, \dots, P_{n_i,n_j})^{n+1}$ are obtained, the auxiliary points may be computed by solving the first $(n_j + 1)$ equations.

Therefore, we have to solve linear system (24). Dividing the first row by $\sum_{j=0}^{n_j} (k_{r0,j}/c_{0,j})$, this system may be expressed as

$$(I + \theta T) \mathbf{P}^{n+1} = (I - (1-\theta)T) \mathbf{P}^n + \psi, \quad (25)$$

where matrix $T \in R^{\tilde{N} \times \tilde{N}}$ ($\tilde{N} = (n_j + 1) \cdot n_i + 1$) is

$$\begin{pmatrix} \tau & \tau_{0,0} & \cdot & \cdot & \tau_{0,n_j} & & & & & & \\ c_{1,0} & a_{1,0} & f_{1,0} & & b_{1,0} & & & & & & \\ \cdot & \cdot & \cdot & & \cdot & & & & & & \\ \cdot & \cdot & \cdot & & \cdot & & & & & & \\ c_{1,n_j} & & & g_{1,n_j} & a_{1,n_j} & & b_{1,n_j} & & & & \\ & c_{i,0} & & & a_{i,0} & f_{i,0} & & b_{i,0} & & & \\ & & \cdot & & \cdot & \cdot & & \cdot & & & \\ & & & \cdot & \cdot & \cdot & & \cdot & & & \\ & & & c_{i,n_j} & & g_{i,n_j} & a_{i,n_j} & & & & b_{i,n_j} \\ & & & & c_{n_i,0} & & & a_{n_i,0} & f_{n_i,0} & & \\ & & & & & \cdot & & \cdot & \cdot & & \\ & & & & & & \cdot & \cdot & \cdot & & \\ & & & & & & & c_{n_i,n_j} & g_{n_i,n_j} & a_{n_i,n_j} \end{pmatrix}$$

where

$$\tau = \frac{\sum_{j=0}^{n_j} k_{r0,j} (1 + (\overline{a_{0,j}}/c_{0,j}))}{\sum_{j=0}^{n_j} (k_{r0,j}/c_{0,j})}, \quad \tau_{00} = \frac{(k_{r0,0}/c_{0,0})}{\sum_{j=0}^{n_j} (k_{r0,j}/c_{0,j})} b_{0,0}, \quad \tau_{0,n_j} = \frac{(k_{r0,n_j}/c_{0,n_j})}{\sum_{j=0}^{n_j} (k_{r0,j}/c_{0,j})} b_{0,n_j}.$$

$(I + \theta T)$ is strictly diagonally dominant; therefore, it is not singular, and it follows that

$$\mathbf{P}^{n+1} = (I + \theta T)^{-1}(I - (1 - \theta)T)\mathbf{P}^n + (I + \theta T)^{-1}\psi. \quad (26)$$

Let $M = (I + \theta T)^{-1}(I - (1 - \theta)T)$. In order to analyze the stability of the finite difference schemes, we prove the following propositions.

PROPOSITION 1. *There is a basis of $R^{\bar{N}}$ ($\bar{N} = (n_j + 1) \cdot n_i + 1$) formed by eigenvectors of M , and all of M 's eigenvalues are real.*

PROOF. Since $M = (I + \theta T)^{-1}(I - (1 - \theta)T)$, if λ_k is an eigenvalue of T , with corresponding eigenvector v_k , then $\beta_k = (1 - (1 - \theta)\lambda_k)/(1 + \theta\lambda_k)$ is an eigenvalue of M with the same eigenvector [12]. (Let us note that $(1 + \theta\lambda_k)$ are the eigenvalues of the nonsingular matrix $(I + \theta T)$, so they are different from zero.) Therefore, we have to prove Proposition 1 for matrix T . Coefficients of matrix T satisfy the following properties:

$$\begin{aligned} c_{i+1,j} &= e^{-2\Delta x} \frac{\beta_{i,j}}{\beta_{i+1,j}} b_{i,j}, & g_{i,j+1} &= \frac{\beta_{i,j}}{\beta_{i,j+1}} f_{i,j}, \\ \frac{k_{r_{0,j}}}{c_{0,j}} b_{0,j} &= \frac{k_{r_{0,j}}}{-(\mu c \Delta t k_{r_{0,j}} / \beta_{0,j} r_w^2 \Delta x^2 \mu B_0)} b_{0,j} = -\frac{r_w^2 \Delta x^2 B_0}{c \Delta t} \beta_{0,j} b_{0,j} = \varepsilon \beta_{0,j} b_{0,j}, & \varepsilon < 0. \end{aligned}$$

We define a diagonal matrix $D \in R^{\bar{N} \times \bar{N}}$ as $d_{0,0} = x_{0,0}$ and $d_{k,k} = x_{i,j}$; $k = (n_j + 1)(i - 1) + j + 1$ ($(j = 0, n_j); i = 1, n_i$); where

$$x_{0,0} = 1 \quad \text{and} \quad x_{i,j} = \sqrt{\frac{\varepsilon \beta_{i,j}}{\sum_{l=0}^{n_j} (k_{r_{0,l}} / c_{0,l})}} e^{i\Delta x} \quad ((j = 0, n_j); i = 1, n_i)$$

($x_{i,j}$ are properly defined because $\varepsilon < 0$, and $\sum_{l=0}^{n_j} (k_{r_{0,l}} / c_{0,l}) < 0$ because $c_{0,l} < 0$ for all l).

It is easily observed that DTD^{-1} is a symmetric matrix. Consequently, DTD^{-1} has real eigenvalues and a basis formed by eigenvectors of DTD^{-1} exists. But DTD^{-1} and T are similar matrices [13], so the same conclusion applies to T and the proposition is proven.

PROPOSITION 2. *Every eigenvalue λ_k of T lies in the segment $[0, 4(\Delta t / \phi_{\min})(k_{r_{\max}} / r_w^2 \Delta x^2 + k_{z_{\max}} / H^2 \Delta z^2)]$, where $k_{r_{\max}}$ and $k_{z_{\max}}$ are upper bounds for horizontal and vertical permeability, respectively, and ϕ_{\min} is a lower bound for porosity.*

PROOF. The statement is immediately proven by applying the Gershgorin's Circle Theorem [12].

PROPOSITION 3. *Let β_k be the eigenvalues of $M = (I + \theta T)^{-1}(I - (1 - \theta)T)$. Therefore,*

- (a) *if $1/2 \leq \theta \leq 1$ then $|\beta_k| \leq 1$ for all $k, \Delta x, \Delta t, \Delta z$;*
- (b) *if $0 \leq \theta < 1/2$ then $|\beta_k| \leq 1$ for all k , if $\Delta t, \Delta x$ and Δz are chosen so that*

$$\Delta t(k_{r_{\max}} / r_w^2 \Delta x^2 + k_{z_{\max}} / H^2 \Delta z^2) \leq (\phi_{\min} / 2(1 - 2\theta)).$$

PROOF. $\beta_k = (1 - (1 - \theta)\lambda_k)/(1 + \theta\lambda_k) = 1 - \lambda_k/(1 + \theta\lambda_k)$, where λ_k are the eigenvalues of matrix T . Let us note that since T is a singular matrix, $\lambda_k = 0$ is an eigenvalue of T , and therefore $\beta_k = 1$ is an eigenvalue of M . Now, we analyze eigenvalues different from unity, i.e., $\lambda_k \neq 0$. In that case, $\beta = 1 - 1/(\lambda^{-1} + \theta)$. From Proposition 2, $\lambda > 0$, so $\beta < 1$. Moreover,

- (a) *if $1/2 \leq \theta \leq 1$, it follows that $\lambda^{-1} + \theta > 1/2$, so that $1 - 1/(\lambda^{-1} + \theta) > -1$ for all $\Delta x, \Delta t, \Delta z$;*
- (b) *if $0 \leq \theta < 1/2$ and $\Delta t(k_{r_{\max}} / r_w^2 \Delta x^2 + k_{z_{\max}} / H^2 \Delta z^2) \leq (\phi_{\min} / 2(1 - 2\theta))$, it follows that*

$$\begin{aligned} \lambda^{-1} + \theta &\geq \frac{\phi_{\min}}{4\Delta t((k_{r_{\max}} / r_w^2 \Delta x^2) + (k_{z_{\max}} / H^2 \Delta z^2))} + \theta \\ &\geq \frac{\phi_{\min}}{(4\phi_{\min} / 2(1 - 2\theta))} + \theta = \frac{1 - 2\theta}{2} + \theta = \frac{1}{2}, \end{aligned}$$

so that $1 - 1/(\lambda^{-1} + \theta) \geq -1$.

Therefore, in (a) and (b) we obtain $|\beta_k| \leq 1$ for all k , and the proposition is proven.

Finally, a suitable norm is defined. Let $\{\mathbf{v}_k\}_{k=1, \bar{N}}$ be a basis of $R^{\bar{N}}$ formed by eigenvectors of M , and $\{\beta_k\}_{k=1, \bar{N}}$ the corresponding eigenvalues. If $\mathbf{x} \in R^{\bar{N}}$, $\mathbf{x} = \sum_{k=1}^{\bar{N}} x_k \mathbf{v}_k$, and we define $\|\mathbf{x}\| = \Delta x \Delta z \sum_{k=1}^{\bar{N}} |x_k|$. We take the matrix norm subordinate to this vector norm, i.e.,

$$\|M\| = \max \frac{\|M\mathbf{x}\|}{\|\mathbf{x}\|}, \quad \|\mathbf{x}\| \neq 0. \quad (27)$$

PROPOSITION 4.

- (a) If $1/2 \leq \theta \leq 1$, then $\|M\| \leq 1$, for all $\Delta x, \Delta z, \Delta t$. Therefore, the finite difference scheme defined in equation (16) is unconditionally stable [14].
- (b) If $0 \leq \theta < 1/2$ and $\Delta t(k_{r_{\max}}/r_w^2 \Delta x^2 + k_{z_{\max}}/H^2 \Delta z^2) \leq (\phi_{\min}/2(1 - 2\theta))$, then $\|M\| \leq 1$. Therefore, the finite difference scheme defined in equation (16) is stable if $\Delta x, \Delta z, \Delta t$ are selected so that $\Delta t(k_{r_{\max}}/r_w^2 \Delta x^2 + k_{z_{\max}}/H^2 \Delta z^2) \leq (\phi_{\min}/2(1 - 2\theta))$.

PROOF. It follows immediately from definition of matrix norm, equation (27), and Proposition 3.

PROPOSITION 5. (Convergence). Let $p(x, z, t)$ be the solution of the differential equation (15) with initial and boundary conditions given by equations (9)–(13) and let $p_{i,j}^n = p(i\Delta x, j\Delta z, n\Delta t)$. Let $P_{i,j}^n$ be the solution of the difference system (25). If T_x and T_z are twice boundedly differentiable, $p_{xxx}, p_{zzz}, p_{ttt}, p_{ttx}, p_{ttz}$ are bounded and $\Delta x, \Delta z$ are chosen as $O(\Delta t^{1/2})$, it may be shown that,

$$\|\mathbf{p}^n - \mathbf{P}^n\| \rightarrow 0, \quad \text{if } \Delta t \rightarrow 0.$$

PROOF. This proof follows the ideas given by Douglas [4]. Under the hypothesis stated above, it may be shown that \mathbf{p}^n satisfies the following equation:

$$(I + \theta T)\mathbf{p}^{n+1} = (I - (1 - \theta)T)\mathbf{p}^n + \psi + \mathbf{h}_n,$$

with $\|\mathbf{h}_n\| \leq h = O(\Delta t^{3/2})$; \mathbf{h}_n is the truncation error resulting from replacing derivatives by finite differences. Let us define $\mathbf{v}^n = \mathbf{p}^n - \mathbf{P}^n$. Therefore, \mathbf{v}^n satisfies $(I + \theta T)\mathbf{v}^{n+1} = (I - (1 - \theta)T)\mathbf{v}^n + \mathbf{h}_n$ and $\mathbf{v}^0 = 0$ (initial condition), i.e.,

$$\mathbf{v}^{n+1} = (I + \theta T)^{-1}(I - (1 - \theta)T)\mathbf{v}^n + (I + \theta T)^{-1}\mathbf{h}_n = M\mathbf{v}^n + \bar{\mathbf{h}}_n, \quad \text{and } \mathbf{v}^0 = 0.$$

Then,

$$\mathbf{v}^n = M^n \mathbf{v}^0 + \sum_{j=0}^{n-1} M^j \bar{\mathbf{h}}_{n-j-1} = \sum_{j=0}^{n-1} M^j \bar{\mathbf{h}}_{n-j-1}, \quad \text{so that } \|\mathbf{v}^n\| \leq \sum_{j=0}^{n-1} \|M\|^j \|\bar{\mathbf{h}}_{n-j-1}\|.$$

From Proposition 4, $\|M\| \leq 1$. Moreover,

$$\|\bar{\mathbf{h}}_k\| = \|(I + \theta T)^{-1}\mathbf{h}_k\| \leq \|(I + \theta T)^{-1}\| \|\mathbf{h}_k\| \leq \|(I + \theta T)^{-1}\| h.$$

But $\|(I + \theta T)^{-1}\| \leq 1$, because 1 is the maximum eigenvalue of $(I + \theta T)^{-1}$ (Proposition 2). Finally,

$$\|\mathbf{p}^n - \mathbf{P}^n\| \leq \|\mathbf{v}^n\| \leq n \cdot h \leq \frac{t_{\max}}{\Delta t} \cdot h = \frac{t_{\max}}{\Delta t} O(\Delta t^{3/2}) \rightarrow 0, \quad \text{if } \Delta t \rightarrow 0.$$

5. LINEAR EQUATIONS SOLUTION AT EACH TIME STEP

In order to solve equation (25), an iterative technique, the Block Successive Over Relaxation method (BSOR) [1,15] is applied. Let us notice that $(I + \theta T)$ is almost pentadiagonal, except for the first row and column, with the following structure:

$$\begin{pmatrix} A_1 & B_1 & & & & \\ C_2 & A_2 & B_2 & & & \\ & \dots & \dots & \dots & & \\ & & \dots & \dots & \dots & \\ & & & \dots & \dots & \dots \\ & & & & C_{n_i-1} & A_{n_i-1} & B_{n_i-1} \\ & & & & & C_{n_i} & A_{n_i} \end{pmatrix},$$

where

$$A_1 = \begin{pmatrix} \alpha & \gamma_0 & \cdot & \gamma_{n_j} \\ c_{1,0} & & & \\ \cdot & \hat{A}_1 & & \\ c_{1,n_j} & & & \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & \cdot & 0 \\ & \hat{B}_1 & \end{pmatrix},$$

$\hat{A}_1, A_2, \dots, A_{n_i}$ are tridiagonal matrices $\in R^{(n_j+1) \times (n_j+1)}$ and $\hat{B}_1, B_2, \dots, B_{n_i-1}; C_2, \dots, C_{n_i}$ are diagonal matrices of the same order.

This method departs from an initial iterate $\mathbf{u}^\nu = (u_0^\nu, \mathbf{u}_1^\nu, \dots, \mathbf{u}_{n_i}^\nu)$, where \mathbf{u}_k^ν , $k = 1, n_i$ are vectors in $R^{(n_j+1)}$ and u_0^ν is a scalar. The new iterate is obtained as follows:

1. Compute $(u_0^{\nu+1}, \mathbf{u}_1^{\nu+1})$ solving

$$\begin{pmatrix} \alpha & \gamma_0 & \cdot & \gamma_{n_j} \\ c_{1,0} & & & \\ \cdot & \hat{A}_1 & & \\ c_{1,n_j} & & & \end{pmatrix} \begin{pmatrix} u_0^{\nu+1} \\ \mathbf{u}_1^{\nu+1} \end{pmatrix} = \begin{pmatrix} \sigma_0 \\ \sigma_1 \end{pmatrix} - \begin{pmatrix} 0 \\ \hat{B}_1 \mathbf{u}_2^\nu \end{pmatrix}.$$

To do this we find a root of the single variable function, $F(\xi) = \sigma_0 - \alpha\xi - \gamma^\top \mathbf{u}_1(\xi)$, where $\mathbf{u}_1(\xi)$ is the solution of $\hat{A}_1 \mathbf{u}_1 = \sigma_1 - \hat{B}_1 \mathbf{u}_2^\nu - \xi \mathbf{c}_1$ (tridiagonal system).

2. Solve each tridiagonal system step by step as

$$A_k \mathbf{u}_k^{\nu+1} = -C_k \mathbf{u}_{k-1}^{\nu+1} - B_k \mathbf{u}_{k+1}^\nu + \sigma_k.$$

The iteration process continues until convergence is obtained.

6. NUMERICAL SIMULATIONS

We divide the analysis in two parts. In the first part, the 2-D simulator is tested by comparing its results against those of the analytical solution of the 1-D case [7]. We consider a well with radius $r_w = 0.1$ m located in the central axis of a cylindrical reservoir with an outer radius $r_e = 500$ m and a thickness $H = 15$ m. The reservoir is formed by four layers of equal thickness. In order to compare with the 1-D analytical solution, it is assumed that all layers have the same properties and that there is no interlayer crossflow. Therefore, horizontal permeability and porosity are constant, while vertical permeability is nil. Data are shown in Table 1. With these assumptions, each layer must behave in the same way with total flow rate equally divided among them. Our purpose is to test this behavior to see if it coincides with the results obtained using the analytical solution.

In the second step, we assume that the four layers have different properties and there is interlayer crossflow. Oil has the same properties shown in Table 1, and permeability and porosity

Table 1. Rock and fluid properties.

porosity, ϕ	0.2
horizontal permeability, k_r	$0.1 \mu\text{m}^2$
vertical permeability, k_z	$0 \mu\text{m}^2$
oil viscosity, μ	0.01 Pa.s
oil compressibility, c	$1.45 \text{ E}-9 \text{ 1/Pa}$
total flow rate, q	$0.001 \text{ m}^3/\text{s}$

Table 2. Rock properties of each layer.

	Layer 1	Layer 2	Layer 3	Layer 4
porosity, ϕ	0.2	0.2	0.2	0.2
horizontal permeability, k_r	$0.010 \mu\text{m}^2$	$0.500 \mu\text{m}^2$	$0.050 \mu\text{m}^2$	$0.100 \mu\text{m}^2$
vertical permeability, k_z	$0.001 \mu\text{m}^2$	$0.050 \mu\text{m}^2$	$0.005 \mu\text{m}^2$	$0.010 \mu\text{m}^2$

of each layer may be seen in Table 2. (Porosity is constant and vertical permeability is 10% of horizontal permeability.)

In both cases we take $n_i = 20$ and $n_j = 3$, i.e., $\Delta x = 0.426$ and $\Delta z = 0.25$ (dimensionless). Besides, gravity effects are neglected.

All simulations have been performed using a 486 DX II personal computer with 4 MB of RAM memory and 66 MHz.

Comparison with Analytical Solution

Different finite difference schemes (corresponding to different θ values, equation (16)) are tested with the analytical solution. As expected, using the 2-D model, all layers have the same behavior, and this behavior coincides with the analytical solution. This may be observed in Figure 2, where the pressure at the well (p_{wf}) is plotted as a function of time. Moreover, pressure profiles computed with different θ values are very similar.

Figure 3 shows relative errors defined as

$$E_r(t_i) = \frac{|p_{wf}^{\text{an}}(t_i) - p_{wf}^{2\text{D}}(t_i)|}{|p_{wf}^{\text{an}}(t_i)|}, \quad (28)$$

where superscripts “an” and “2D” mean analytical solution and 2-D model solution, respectively. Relative errors are very low and almost equal for different θ values. Nevertheless, at early times, errors decrease when θ increases, as Figure 3 illustrates.

Therefore, we choose $\theta = 1$ to test the BSOR technique. This unconditionally stable scheme is selected in order to analyze how much the time step Δt can be increased without distorting the goodness of the numerical approximation. BSOR reaches convergence even with large values of time increments. We test $\Delta t = 0.1, 1, 10, 100$ s. Relative errors, obtained using these different time increments, are shown in Figure 4. In each case, the numerical approximation is not good until a certain value of time is reached. It is interesting to note that, as time increases, relative errors tend to a constant value, independently of Δt . Nevertheless, the CPU time required to complete each time step increases when Δt increases, as Table 3 shows.

Table 3. Average CPU time required for BSOR to complete each time step.

Δt (s)	CPU time (s)
0.1	0.092
1	0.329
10	1.317
100	38.4

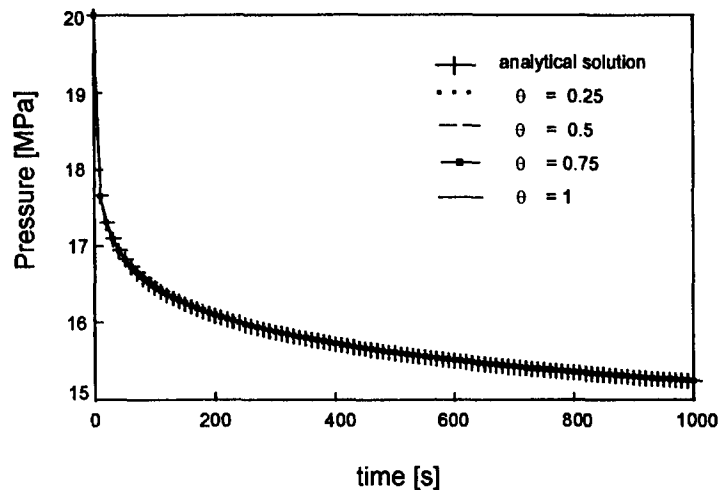


Figure 2. Comparison with analytical solution using different schemes.

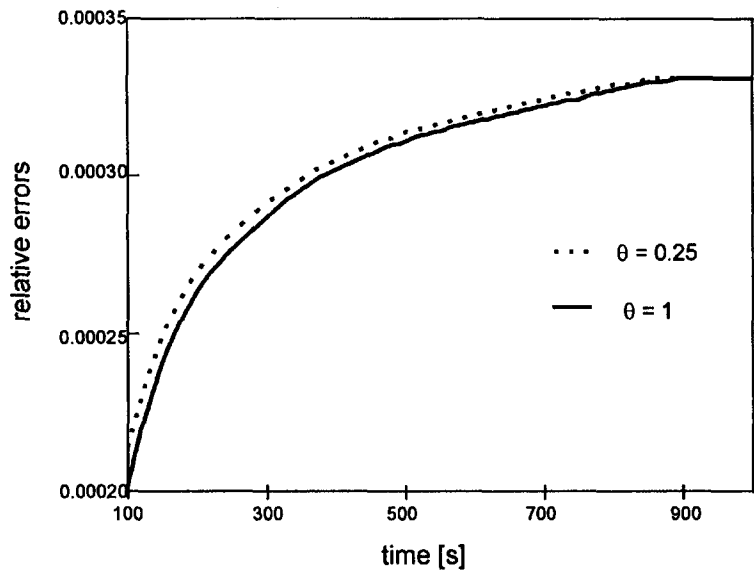


Figure 3. Relative errors for the numerical solution using different schemes.

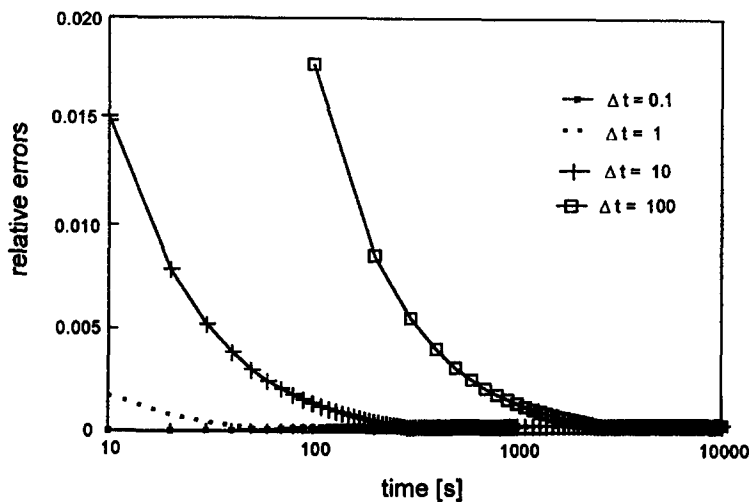


Figure 4. Relative errors using different time increments.

Our conclusion is that the best choice is to take a variable time increment algorithm, using an appropriate Δt for each range of time values. Therefore, we use a variable time increment in the following section.

Simulation of an Interlayer Crossflow Case

Now, the 2-D model is used to simulate a problem with interlayer crossflow among layers of different characteristics. Fluid data are taken from Table 1 and rock properties from Table 2. We apply the BSOR technique taking $\theta = 1$ in two cases: (a) with a fixed $\Delta t = 0.01$ s and (b) with a variable time step, selected in the following way:

- in time interval $[0, 1]$ [s], we take $\Delta t = 10^{-3}$ s,
- in time interval $[10^k, 10^{k+1}]$ [s], we take $\Delta t = 10^{k-2}$ s.

Pressures computed with fixed and variable Δt are shown in Figure 5. The agreement between them is excellent, and the corresponding CPU time decreases from 1.73 hours to 15 minutes.

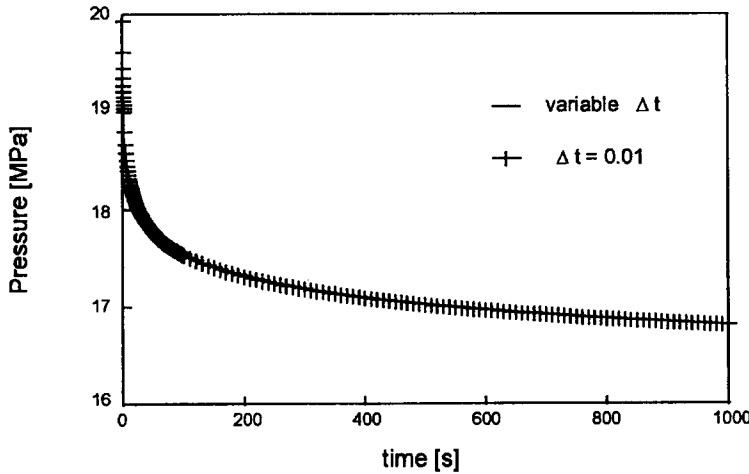


Figure 5. Crossflow case: comparison between fixed and variable time increment.

7. CONCLUSIONS

The equation that models the single phase flow of oil towards a well in cylindrical coordinates was solved in 1-D (coordinate r) and 2-D (coordinates (r, z)), using a family of finite difference schemes. This family depends on one parameter θ . We studied the stability of these schemes, taking into account the boundary conditions. Moreover, some numerical experiments are presented in order to test the proposed numerical schemes. BSOR iterative method for solving linear systems is used, specifically implemented for this particular problem. Conclusions are the following.

1. For the 2-D model, if $1/2 \leq \theta \leq 1$, the finite difference scheme defined in equation (16) is unconditionally stable. If $0 \leq \theta < 1/2$, the scheme is stable if $\Delta x, \Delta z, \Delta t$ are selected so that $\Delta t(k_{r_{\max}}/r_w^2 \Delta x^2 + k_{z_{\max}}/H^2 \Delta z^2) \leq \phi_{\min}/2(1 - 2\theta)$.
2. Conclusion 1 holds for the 1-D model. For $0 \leq \theta < 1/2$, the stability condition becomes $(\Delta t/\Delta x^2) \leq \phi_{\min} r_w^2 / k_{r_{\max}} 2(1 - 2\theta)$.
3. The finite difference solutions verify the analytical solution of a simplified 1-D model.
4. BSOR converges using large values of Δt , so that this method is recommended when a long period of time must be simulated. Besides, the CPU time is reduced using a variable time increment.
5. The 2-D model presented here may be used to analyze the influence of vertical permeability, vertical and radial heterogeneities and also gravity effects on well test pressure response.

REFERENCES

1. K. Aziz and A. Settari, *Petroleum Reservoir Simulation*, Elsevier Applied Science Publishers, London, (1985).
2. D. Peaceman, *Fundamentals of Numerical Reservoir Simulation*, Elsevier, New York, (1977).
3. R. Home, *Modern Well Test Analysis—A Computer Aided Approach*, Petroway, Palo Alto, CA, (1990).
4. J. Douglas, Jr., The application of stability analysis in the numerical solution of quasi-linear parabolic differential equations, *Trans. Amer. Math. Soc.* **89**, 484–518 (1958).
5. Y. Cherruault, M. Choubane, J. Valleton and J. Vincent, Stability and asymptotic behavior of a numerical solution corresponding to a diffusion-reaction equation solved by a finite difference scheme (Crank-Nicolson), *Computers Math. Applic.* **20** (11), 37–46 (1990).
6. R. Lardner, Stability of the numerical solution of a parabolic system with integral subsidiary conditions, *Computers Math. Applic.* **19** (12), 41–46 (1990).
7. G. Savioli, M. Goldschmit and M.S. Bidner, Discusión sobre las soluciones analíticas y numéricas de la ecuación radial de difusividad que representa el flujo en medios porosos, *Revista Brasileira de Engenharia* **5** (2), 65–79 (1988).
8. D. Oliver, The averaging process in permeability estimation from well test data, *Society of Petroleum Engineers Formation Evaluation* **5** (3), 319–326 (1990).
9. R. Prijambodo, R. Raghavan and A. Reynolds, Well test analysis for wells producing layered reservoirs with crossflow, *Society of Petroleum Engineers Journal* **25** (3), 380–396 (1985).
10. M.L. Katz and M.R. Tek, A theoretical study of pressure distribution and fluid flux in bounded stratified porous systems with crossflow, *Society of Petroleum Engineers Journal*, 68–82 (March 1962).
11. G. Savioli, M.S. Bidner and P. Jacovkis, The influence of heterogeneities on well test pressure response—A sensitivity analysis, *Society of Petroleum Engineers Advanced Technology Series* **4** (1), 67–72 (1996).
12. G. Smith, *Numerical Solution of Partial Differential Equations, Finite Difference Methods*, 3rd edition, Oxford University Press, Oxford, (1985).
13. G. Golub and C. Van Loan, *Matrix Computations*, The Johns Hopkins University Press, Baltimore, MD, (1984).
14. R. Ritchmyer and K. Morton, *Difference Methods for Initial-Value Problems*, 2nd edition, Interscience, New York, (1967).
15. R. Varga, *Matrix Iterative Analysis*, Prentice Hall, New Jersey, (1962).